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Some applications of differential topology in general relativity

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Recently, there have been several applications of differential and algebraic topology to problems concerned with the global structure of spacetimes. In this paper, we derive obstructions to the existence of spin-Lorentz and pin-Lorentz cobordisms and we show that for compact spacetimes with non-empty boundary there is no relationship between the homotopy type of the Lorentz metric and the causal structure. We also point out that spin-Lorentz and tetrad cobordisms are equivalent. Furthermore, because the original work on metric homotopy and causality may not be known to a wide audience, we present an overview of the results here.

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1. Definitions and conventions

In this paper, by the word "spacetime" we shall mean a four-dimensional manifold M, connected and smooth (though not necessarily orientable), possessing an everywhere non-singular Lorentz metric g_{ab}.

The existence of an everywhere non-singular Lorentz metric on a time-orientable M is equivalent to the existence of a global non-vanishing vector field v. To see this, recall that the underlying Riemannian manifold M possesses a Riemannian metric, g^{R}_{ab}. Given a vector field v, one can define the Lorentz metric, g_{ab}, in terms of the Riemannian metric and v via the relation

$$g_{ab} = g^{\mathrm{R}}_{ab} - 2v_a v_b / (g^{\mathrm{R}}_{ab} v^a v^b) . \qquad (1)$$

The converse follows by diagonalising the given Lorentz metric into "Riemannian metric" and (negative eigenvalue) "eigenvector" parts, and defining v to be the vector with negative eigenvalue (see [1] or [2]).

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We assume [#1] (for the time being) that our spacetimes are time-orientable, i.e. that we can make a globally consistent choice for the sign of v (one cannot propagate v around some closed loop in M and end up with $-v$).

Broadly speaking, the kink number is an integer which classifies metrics up to homotopy. To make this more precise, let (M, g_{ab}) be a spacetime and $\Sigma \subset M$ a three-dimensional, connected, orientable submanifold. Since Σ is three-dimensional and oriented, we can always find a global framing $\{u_i : i = 1, 2, 3\}$ of Σ together with a unit normal, n, to Σ. We can then extend this tetrad framing (n, u_i) of Σ to a collar neighbourhood

$$N \cong \Sigma \times [0, 1].$$

(We extend to N to deal with the case $\Sigma \cong \partial M$). Let v be the unit timelike vector determined by g_{ab}; then v can be written

$$v = v^0 n + v^i u_i \tag{2}$$

such that $\sum_i (v^i)^2 = 1$. Clearly, then, v determines a map $K : \Sigma \longrightarrow S^3$, by assigning to each point $p \in \Sigma$ the direction in $T_p M$ (a point on the S^3 determined by the tetrad (n, u_i)) that v_p points to, i.e., as visualised in fig. 1. The north pole of S^3 is given by n. We then define the kink number of g_{ab} with respect to Σ as

$$\text{kink}(\Sigma; g_{ab}) = \deg(K) \tag{3}$$

where $\deg(K)$ is "the degree of the mapping K".

Convention 1. If v is a timelike vector determined by g_{ab}, we shall often write

$$\text{kink}(\Sigma; g_{ab}) = \text{kink}(\Sigma; v).$$

Now, for our immediate purposes we shall be concerned with kinking with respect to ∂M, the boundary of our spacetime. In particular, we shall be concerned with the case M compact, with

$$\partial M \cong \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_n \tag{4}$$

where the Σ_i's are now closed, connected, oriented three-manifolds and $\cup$ is the operation of disjoint union. We wish to define the quantity $\text{kink}(\partial M; g_{ab})$. On differential topological grounds (see [3]) we see that it makes sense to write

$$\text{kink}(\partial M; g_{ab}) = \sum_i \text{kink}(\Sigma_i; g_{ab}) \tag{5}$$

once we have decided on some convention for choosing the sign of n_i (the unit normal to each Σ_i) consistently. Our convention is simply that n_i is always

[#1] Note: It is not necessary (generally) to assume time-orientability; we could still define a notion of kinking for non-orientable M.

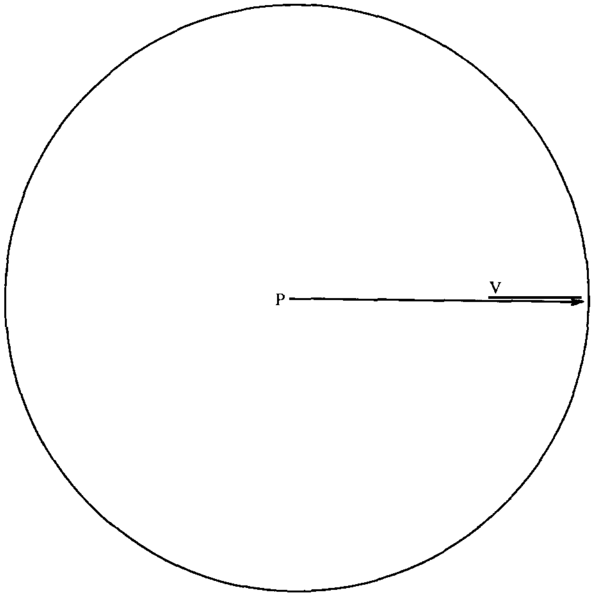

Fig. 1. The three-sphere is the set of directions (at a point) in the four-manifold.

pointing *out* of M. Having established this, we can now discuss the concept of cobordism. First, however, we recall the following

Definition 2. Let M be an oriented and time-orientable spacetime, with orthonormal frame bundle $O(M)$ a principal bundle with structure group $SO(1,3)_0$. We say that M has $SL(2,\mathbb{C})$*-spin structure* iff there exists a principal bundle $\overline{O}(M)$ (with structure group $\mathrm{Spin}(1,3)_0 \simeq SL(2,\mathbb{C})$) which is a 2–1 covering of $O(M)$, so that the following diagram commutes:

$$\begin{array}{ccccc} SL(2,\mathbb{C}) \simeq \mathrm{Spin}(1,3)_0 & \longrightarrow & \overline{O}(M) & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \text{ identity} \\ SO(1,3)_0 & \longrightarrow & O(M) & \longrightarrow & M \end{array}$$

We shall call such an M a *spin-Lorentz manifold.*

Now, suppose we are given a collection of three-dimensional, connected, orientable, closed manifolds $\Sigma_1, \Sigma_2, ..., \Sigma_n$. We say that there is a *spin-Lorentz cobordism* for $\{\Sigma_i : i = 1, ..., n\}$ iff there exists a spin-Lorentz manifold M satisfying

$$\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$$

We have the related

Definition 3. Let M be a smooth, four-dimensional Riemannian manifold. We say that M is *parallelisable* iff there exists a global non-vanishing tetrad field, $\{e_i\}$, on M.

Let $\Sigma_1, ..., \Sigma_n$ be a collection of three-manifolds as above. We say that there is a *tetrad cobordism* for $\{\Sigma_i : i = 1, ..., n\}$ iff there exists a parallelisable four-manifold M such that

$$\partial M \cong \Sigma_1 \cup \Sigma_2 \cup ... \cup \Sigma_n.$$

More generally, we can consider the problem of finding cobordisms admitting other types of structures; the study of this problem, in the general setting, has been extensively developed (see [4]). Note, finally, that it is not necessary for a spacetime M to be time-orientable in order to "mimic" the above constructions in a sensible way. For non-time-orientable M we still have a notion of kinking [#2] (defined now as the degree of the map from Σ to $\mathbb{RP}^3$) and we still have a notion of "pinors", defined thus:

Definition 4. Let M be a non-orientable spacetime, with orthonormal frame bundle $O(M)$ a principal bundle with structure group $O(p, q)$. We say that M has *pin-Lorentz structure* iff there exists a principal bundle $\overline{O}(M)$ (with structure group $\mathrm{Pin}(p, q)$) which is a 2–1 covering of $O(M)$, so that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Pin}(p,q) & \longrightarrow & \overline{O}(M) & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \text{ identity} \\ O(p,q) & \longrightarrow & O(M) & \longrightarrow & M \end{array}$$

where either (i) $p = 1, q = 3$, or (ii) $p = 3, q = 1$.

As we shall see, the topological obstruction to pin-Lorentz structure is related to that of spin-Lorentz structure.

2. Equivalence of tetrad and spin-Lorentz cobordism

One of the first questions that comes to mind is whether or not there is any connection between tetrad and spin-Lorentz cobordism. That there should be some relation is implied by a theorem of Geroch [5]. One simple approach to this question would be to calculate the topological obstruction to tetrad cobordism and compare it to the obstruction to spin-Lorentz cobordism. However, this is not necessary because we have the following

Theorem 5 (Hirzebruch and Hopf [6]). *Let M be a smooth, compact, orientable four-manifold. Then M is parallelisable iff $p_1(M) = 0, e(M) = 0$, and $w_2(M)$*

[#2] Likewise, the existence of a globally non-singular Lorentz metric on a non-time-orientable spacetime is now equivalent to the existence of a global non-vanishing *line field* $\{v, -v\}$.

$= 0$, *where* $p_1(M)$ *is the first Pontryagin number,* $e(M)$ *is the Euler number, and* $w_2(M)$ *is the second Stiefel–Whitney class.*

Now, when M has non-trivial boundary $\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n \neq \emptyset$, $w_2(M)$ is defined as usual, but the relationship betwen $e(M)$ and the zeros of smooth vector fields on M changes [3]. That is, if v is a smooth vector field on M, and $\sum i_v$ denotes "the sum of the indices of v", then $e(M)$ is given by

$$\sum i_v = e(M) + \text{kink}(\partial M; v) \tag{6}$$

Furthermore, for a manifold with non-empty boundary (of disjoint closed, orientable three-manifolds) we automatically have

$$p_1(M) = 0.$$

Thus, amending the above theorem to deal with the case when $\partial M \neq \emptyset$, we obtain

Corollary 6. *Let M be a smooth, compact orientable four-manifold with non-empty boundary*

$$\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n \neq \emptyset.$$

Then M is parallelisable by a tetrad field $\{v_i : i = 1, \ldots 4\}$ *iff* $w_2(M) = 0$ *and* $\sum i_{v_i} = 0$, *for any vector* v_i *in the tetrad.*

Now, notice that if v is a timelike vector field on M with respect to a Lorentz metric g_{ab}, then the metric is globally non-singular iff

$$\sum i_v = 0.$$

Furthermore, M admits a spin structure iff $w_2(M) = 0$. Thus, the obstructions to (M, g_{ab}) being a spin-Lorentz manifold are precisely the obstructions to M being parallelisable, and so spin-Lorentz and tetrad cobordism are equivalent.

3. Derivation of the obstruction to spin-Lorentz cobordism

Since we are concerned with the obstruction to spin-Lorentz cobordism, it is useful to first review the second Stiefel–Whitney class $w_2(M)$ (the obstruction to spin structure on M).

Hence, suppose we are given a four-dimensional orientable manifold M with tangent bundle TM. Given the 2–1 covering map

$$\rho : \text{Spin}(4) \longrightarrow SO(4)$$

we can define, for transition function $h_{ab} \in SO(4)$, the lifting $\overline{h}_{ab} \in \text{Spin}(4)$, satisfying

$$\rho(\overline{h}_{ab}) = h_{ab}, \qquad \overline{h}_{ab} = \overline{h}_{ab}{}^{-1}.$$

By local triviality, such a lifting can always be found (locally).

Because ρ is a homomorphism, and using the compatibility condition, we see that

$$\rho(\overline{h}_{ab}\,\overline{h}_{bc}\,\overline{h}_{ca}) = h_{ab}\,h_{bc}\,h_{ca} = \mathrm{Id},$$

where Id is identity map on $U_a \cap U_b \cap U_c \subset M$ (U_is are open sets). Thus, $\overline{h}_{ab}\,\overline{h}_{bc}\,\overline{h}_{ca}$ is in the kernel of ρ_j; however, there is a sign ambiguity in the kernel:

$$\ker = \{\,\pm\mathrm{Id}\,\}. \tag{7}$$

However, for the $\overline{h}_{ab}$'s to define a global spin bundle over M, they must also satisfy the compatibility condition

$$\overline{h}_{ab}\,\overline{h}_{bc}\,\overline{h}_{ca} = \mathrm{Id} \tag{8}$$

Hence, define the Čech 2-cochain

$$w_2(M) \equiv w_2(M; U_i, U_j, U_k) : \ U_i \cap U_j \cap U_k \longrightarrow \mathbb{Z}_2$$

via the relation

$$\overline{h}_{ab}\,\overline{h}_{bc}\,\overline{h}_{ca} = w_2(M)\ \mathrm{Id} \tag{9}$$

(where $\mathbb{Z}_2$ here is multiplicative). Then $w_2(M) \in H^2(M;\mathbb{Z}_2)$ is called the *second Stiefel–Whitney class.*

Taking $\mathbb{Z}_2$ to be additive we have the easy

Lemma 7. *Let M be as above. Then there exists a spin bundle over M iff*

$$w_2(M) = 0.$$

Clearly, if M admits Spin(4) spin structure and M is Lorentz, then M admits $SL(2,\mathbb{C})$ spin structure and is a spin-Lorentz manifold.

To see how we can obtain a topological obstruction to spin-Lorentz structure on M, which depends only on boundary data defined on ∂M, recall the following

Lemma 8 (Milnor and Kervaire [7], p. 517). *Let M be an orientable, smooth manifold of dimension 4. Let $u(\partial M)$ (the mod 2 Kervaire semicharacteristic) be given by*

$$u(\partial M) = \dim_{\mathbb{Z}_2}(H_0(\partial M;\mathbb{Z}_2) \oplus H_1(\partial M;\mathbb{Z}_2)) \bmod 2$$

Then the rank of the intersection pairing $h : H_2(M;\mathbb{Z}_2) \times H_2(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ satisfies $\mathrm{rank}(h) = (u(\partial M) + e(M)) \bmod 2$.

To see how the lemma relates to spin-Lorentz structure, recall [8] that the rank of the intersection pairing also satisfies

$$(\mathrm{rank}(h)) \bmod 2 = 0 \Longleftrightarrow w_2(M) = 0 \tag{10}$$

Combining eq. (10) with the above lemma, we obtain

Theorem 9. *Let $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be a collection of closed, orientable three-manifolds. Then there exists a spin-Lorentz cobordism, M, for $\Sigma_1, \Sigma_2, ..., \Sigma_n$ if and only if*

$$(u(\partial M) + \mathrm{kink}(\partial M; v)) \bmod 2 = 0, \tag{11}$$

where $u(\partial M)$ is as above, and v is the timelike vector determined by the metric on M.

Proof. Suppose such a spin-Lorentz cobordism, M, exists. Then M admits spin structure and so $w_2(M) = 0$; by eq. (10), $(\mathrm{rank}(h)) \bmod 2 = 0$. Hence $(u(\partial M) + e(M)) \bmod 2 = 0$, by the lemma.

Furthermore, since M is a Lorentz manifold with timelike vector v we must have $\sum i_v = 0$ (since v must not vanish); hence, by eq. (6)

$$-e(M) = \mathrm{kink}(\partial M; v)$$

and so, modulo 2, we obtain

$$(u(\partial M) + \mathrm{kink}(\partial M; v)) \bmod 2 = 0.$$

Conversely, suppose that no such spin-Lorentz cobordism exists. Then any cobordism M is one of three things: spin but not Lorentz, Lorentz but not spin, or neither spin nor Lorentz.

If M is spin but not Lorentz, then $w_2(M) = \mathrm{rank}(h) \bmod 2 = 0$ and $\sum i_v \neq 0$. If $\sum i_v$ is even, then we could take the connected sum of M with a finite number of spin manifolds (of even Euler number) to obtain a spin cobordism M' with $\sum i_v = 0$ [1]. However, such an M' would be a spin-Lorentz cobordism, contradicting our assumption. Thus, $\sum i_v$ must be odd and so we get

$$(u(\partial M) + \mathrm{kink}(\partial M; v)) \bmod 2 = 1$$

Likewise, if M is Lorentz but not spin, then $\sum i_v = 0$ and $w_2(M) = (\mathrm{rank}(h)) \bmod 2 = 1$ and so

$$(u(\partial M) + \mathrm{kink}(\partial M; v)) \bmod 2 = 1.$$

Finally, if M is neither spin nor Lorentz, then $\sum i_v \neq 0$ and $w_2(M) = (\mathrm{rank}(h)) \bmod 2 = 1$. If $\sum i_v$ is even, then $(u(\partial M) + \mathrm{kink}(\partial M; v)) \bmod 2 = 1$. Thus, suppose that $\sum i_v$ is odd. Recall that although M is not spin, we can always find [9] a spherical modification of M, M', which is spin (a spin cobordism always exists). However, such a cobordism would satisfy $e(M') \neq e(M) \bmod 2$. Thus, if v' is the vector field v extended to M', we have that $\sum i_{v'}$ is even. However, M' is then a spin manifold with $\sum i_{v'}$ an even number. This case was dealt with above, and we saw that

$$(u(\partial M) + \mathrm{kink}(\partial M; v)) \bmod 2 = 1.$$

This exhausts all possibilities. □

Using section 2 we obtain

Corollary 10. *Let $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be a collection of closed, orientable three-manifolds. Then there exists a tetrad cobordism M, with global tetrad field $\{v_1, v_2, v_3, v_4\}$, if and only if*

$$(u(\partial M) \ + \ \text{kink}(\partial M; v_i)) \bmod 2 = 0$$

for any v_i in the tetrad field.

Thus, we see that spin-Lorentz and tetrad cobordisms between arbitrary three-manifolds always exist, as long as we allow for arbitrary kink number (boundary data).

4. Discussion of Clifford algebras

Before discussing pin structures on non-orientable spacetimes, it is useful to review the Clifford algebras which give rise to the "Cliffordian Pin groups" [#3], and to discuss some of the subtleties associated with these groups.

Thus, let (M, g_{ab}) be any spacetime (not necessarily orientable). Then the tangent bundle of M, τ_M, can always be reduced to a bundle with structure group $O(3,1)$ (for signature $(-+++)$) or $O(1,3)$ (for signature $(+---)$) (actually $O(3,1) \simeq O(1,3)$, but as we shall see it is important that we keep the distinction between the signatures when we pass to the double covers of these groups). Now, we associate to the tangent space of (M, g_{ab}), at some $p \in M$, the "Clifford algebra", $Cl(T_p(M), g_{ab})$, which can be described as follows.

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis (with respect to g_{ab}) for $T_p(M)$. Then $Cl(T_p(M), g_{ab})$ is the algebra generated by $\{e_i | i = 1, ..., 4\}$, subject to the following relation:

$$e_i e_j \ + \ e_j e_i = 2g(e_i, e_j)$$

Now, associated to any Clifford algebra $Cl(p,q)$ is the group of invertible elements, $Cl_*(p,q)$. Let $P(p,q) \subseteq Cl_*(p,q)$ be the subgroup generated by non-null vectors $v \in T_p(M)$ (i.e., $g_{ab}v^a v^b \neq 0$). Then $\text{Pin}(p,q) \subseteq P(p,q)$ is the subgroup generated by elements $v \in T_p(M)$ with $g_{ab}v^a v^b = \pm 1$. Thus, any element $x \in \text{Pin}(p,q)$ can be written as some product: $x = v_1 v_2 \cdots v_n$, where all the v_i's are unit spacelike or timelike vectors. But we know that the groups $\text{Pin}(p,q)$ are double covers of the groups $O(p,q)$, and so in some sense the pin groups must "re-express" all of the information contained in the Lorentz group

[#3] Note: There are other 2–1 covers of $O(p,q)$ which do not arise from any Clifford algebra. These give us "non-Cliffordian" pin structures. The obstructions to non-Cliffordian pin structures have been worked out elsewhere [10]. See [11] for an excellent discussion of the different pin groups.

[in fact, they must "re-express" the information in a "simply connected" way, since $\pi_1(O(p,q)) \simeq \mathbb{Z}_2$ and $\pi_1(\mathrm{Pin}(p,q)) \simeq 0$]. In fact, we see how elements of the pin groups represent Lorentz transformations when we recall the following

Fact 11. *Any element of $O(p,q)$ can be represented as a product of reflections across a finite number of (non-null) planes through the origin $O \in T_p(M)$.*

Thus, let $x = v_1 v_2 \cdots v_n$ be any element of $\mathrm{Pin}(p,q)$. For each vector v_i, let $v_i^{\perp}$ denote the plane perpendicular to v_i. Then, for any element $w \in T_p(M)$, the reflection of w about $v_i^{\perp}$ is given as

$$w \longrightarrow w - 2(w \cdot v_i)v_i$$

Hence, we can view $v_1 v_2 \cdots v_n \in \mathrm{Pin}(p,q)$ as a series of reflections about the planes $v_n^{\perp}, v_{n-1}^{\perp}, \ldots, v_1^{\perp}$, i.e., $v_1 v_2 \cdots v_n$ has a natural interpretation as a Lorentz transformation.

Furthermore, we see that to every Lorentz transformation there correspond two distinct elements of $\mathrm{Pin}(p,q)$. For example, if $T \in O(p,q)$ represents time reversal, and e_1 is the (basis) unit timelike vector, then both e_1 and $-e_1$ correspond to T. And so on.

In the next section, we shall concentrate on the cobordism problem for Cliffordian pin bundles, i.e., bundles whose structure group can be obtained from a Clifford algebra (in the way described above). We note, however, that the cobordism problem for *non*-Cliffordian pin structures has been worked out elsewhere [10].

Indeed, these results can perhaps be taken as further evidence that there is no immediate reason why we should insist that our underlying spacetime manifold be orientable; we can do fermionic physics on non-orientable spacetimes using pin bundles (see [12,13], and in particular [14]). This point is especially relevant if we take the "spacetime foam" picture seriously, since there is no *a priori* reason why nature should prefer orientable fluctuations over non-orientable fluctuations.

5. Derivation of the obstructions to Cliffordian pin-Lorentz cobordism

In order to apply the above lemma of Milnor and Kervaire to the derivation of obstructions to pin-Lorentz structure, we first must derive some identities for the Stiefel–Whitney classes $w_1(M)$ and $w_2(M)$ when M is non-orientable, i.e., when $w_1(M) = 1$.

First, recall Wu's formula

$$w_k(M) = \sum_{i+j=k} Sq^i(v_j) \tag{12}$$

where Sq^i is the Steenrod squaring operation [15], and $v_j \in H^j(M)$ is the unique element which satisfies

$$(v_j \smile x)[w] = Sq^j(x)[w], \ \forall\, x \in H^{n-j}(M) \tag{13}$$

where $\smile$ denotes cup product, and $w \in H_4(M)$ is the fundamental homology class [15]. Using eqs. (12) and (13), together with the axioms for Steenrod squaring [15], we obtain:

$$\begin{aligned} w_1(M) &= v_1 && (14)\\ w_2(M) &= Sq^0(v_2) + Sq^1(v_1) + Sq^2(v_0) \\ &= v_2 + v_1 \smile v_1 \\ &= v_2 + w_1 \smile w_1 && (15) \end{aligned}$$

Thus, let $x_2 \in H^2(M;\mathbb{Z}_2)$ be any 2-cochain, then

$$w_2(M) \smile x_2 = v_2 \smile x_2 + (w_1 \smile w_1) \smile x_2$$

and since $v_2 \smile x_2 = x_2 \smile x_2$, we get

$$w_2(M) \smile x_2 = x_2 \smile x_2 + (w_1 \smile w_1) \smile x_2 \tag{16}$$

Now, recall the definition of "intersection pairing" between two cycles $x, y \in H_2(M;\mathbb{Z}_2)$. First, let x_2 and y_2 be the 2-cochains associated with x and y, defined via

$$\begin{aligned} x_2 \frown w &= x \\ y_2 \frown w &= y && (17) \end{aligned}$$

where $\frown$ denotes cap product. Then we define the intersection pairing, $h : H_2(M;\mathbb{Z}_2) \times H_2(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$, via the relation

$$h(x,y) = x \cdot y = (x_2 \smile y_2) \frown w.$$

Now, the question is: How does the parity of the rank of h relate to $w_2(M)$ and $w_1(M)$? To see the answer, suppose rank(h) was even. Then every cycle $x \in H_2(M)$ would have to have self-intersection number zero, i.e.,

$$\begin{aligned} & x \cdot x = 0 \quad \forall\, x \in H_2(M;\mathbb{Z}_2) \quad \Rightarrow \\ & x \cdot x = (x_2 \smile x_2) \frown w = 0, \quad \forall\, x_2 \in H^2(M;\mathbb{Z}_2) \quad \Rightarrow \\ & ((w_2(M) \smile x_2) - [w_1 \smile w_1] \smile x_2) \frown w = 0 \quad \Rightarrow \\ & w_2(M) - w_1 \smile w_1 = 0 && (18) \end{aligned}$$

Conversely, if rank(h) was odd, then $\exists\, x \in H_2(M;\mathbb{Z}_2)$ such that $x \cdot x \neq 0$, and so

$$w_2(M) - w_1 \smile w_1 \neq 0 \tag{19}$$

Combining (18) and (19), we obtain

$$w_2(M) + w_1 \smile w_1 = 0 \iff \operatorname{rank}(h) = 0 \mod 2 \tag{20}$$

Combining (20) with the lemma of Milnor and Kervaire (which still holds, since everything is in $\mathbb{Z}_2$ coefficients), we get

$$(u(\partial M) + \mathrm{kink}(\partial M; g_{ab}))\ \mathrm{mod} 2 = 0 \Longleftrightarrow (w_2(M) + w_1 \smile w_1) = 0 \quad (21)$$

where $\mathrm{kink}(\partial M; v)$ is, again, the degree of the total map $\partial M \longrightarrow \mathbb{RP}^3$ (or S^3 if M is time-orientable) defined by v.

We are now in position to derive our topological obstructions (which depend only upon boundary data, choice of orientation, choice of signature, and behaviour of 1-cocycles under the cup product) using results of Karoubi [16]. We therefore begin by dividing the possible cases according to signature.

Case 1 (signature $(-+++)$). In the case when the signature is $(-+++)$ we get the following result:

First recall that the tangent bundle τ_M decomposes into a direct sum of subbundles,

$$\tau_M \cong \tau^+ \oplus \tau^-$$

where τ^+ is the "spacelike subbundle" and τ^- is the "timelike subbundle" (the terms refer to the behaviour of sections of these bundles with respect to the Lorentz metric g_{ab}).

By elementary axioms [15] we have

$$\begin{aligned} w_1(\tau_M) &= w_1(\tau^+) + w_1(\tau^-) \\ w_2(\tau_M) &= w_2(\tau^+) + w_2(\tau^-) + w_1(\tau^+) \smile w_1(\tau^-) \end{aligned} \quad (22)$$

We shall often use the abbreviations $w_1(\tau^+) = w_1^+, w_1(\tau^-) = w_1^-, w_2(\tau^+) = w_2^+$, etc.

Now, it is a theorem of Karoubi [16] that there is Pin(3,1) structure on M if and only if the following equation holds:

$$w_2^- + w_2^+ + w_1^- \smile w_1^- + w_1^- \smile w_1^+ = 0 \quad (23)$$

Combining eqs. (22) and (23), we thus see that M has Pin(3,1) structure if and only if

$$w_2(M) = w_2(\tau_M) = w_1^- \smile w_1^- \quad (24)$$

Combining eqs. (21), (22), and (24) we then have

Theorem 12. *Let $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be a collection of closed three-manifolds. Then there exists a pin-Lorentz cobordism, M (of signature $(-+++)$), for $\{\Sigma_i : i = 1, ..., n\}$ if and only if the following holds:*

$$(u(\partial M) + \mathrm{kink}(\partial M; g_{ab}))\ \mathrm{mod}\ 2 = 0 \quad \Longleftrightarrow \quad w_1^+ \smile w_1^+ = 0$$

Proof. Suppose such a pin-Lorentz cobordism, M, exists. Then eq. (24) holds. Combining this with eq. (21) we get

$$(u(\partial M) + \mathrm{kink}(\partial M; g_{ab}))\ \mathrm{mod}\ 2 = 0 \quad \Longleftrightarrow \quad w_1^+ \smile w_1^+ = 0$$

The converse is also immediate. □

We interpret theorem 12 in the following subcases and examples (all of which deal with signature $(-+++)$.

(theenumi) First of all, suppose that we want our cobordism to be both space and time-orientable. Then we must have $w_1^+ = 0$ and $w_1^- = 0$, and so $w_1^+ \smile w_1^+ = 0$. Thus, such a cobordism exists if and only if

$$(u(\partial M) + \text{kink}(\partial M; g_{ab})) \bmod 2 = 0.$$

(theenumi) Similarly, if we insist that our cobordism be neither space nor time-orientable, then we have $w_1^+ = 1 = w_1^-$ and so $w_1^+ \smile w_1^+ = 1$. Thus, such a cobordism exists if and only if

$$(u(\partial M) + \text{kink}(\partial M; g_{ab})) \bmod 2 = 1$$

(theenumi) Now, however, suppose that we want our cobordism to be time-orientable but not space-orientable, i.e., $w_1^- = 0$ but $w_1^+ = 1$. Then we run into various subtleties which are caused by the definition of the cup product, $\smile$. To understand this, let us recall how the cup product is defined simplicially.

First, suppose that a_1, b_1 are two 1-cochains; then their cup product is a 2-cochain which may be defined by its action on a singular simplex $S: T^2 \longrightarrow M$. That is, S is a map which imbeds the convex set

$$\{a_1, a_2, a_3 \in \mathbb{R} | a_i \geq 0,\ a_1 + a_2 + a_3 = 1\} = T^2 \subset \mathbb{R}^3,$$

into M (i.e., a tetrahedron is determined by the origin plus three linearly independent points a_1, a_2, and a_3 in $\mathbb{R}^3$).

Next, let $f(a_1, a_2, a_3) = (a_1, a_2, 0)$ denote the "front 1-face of T^2" (i.e., $(a_1, a_2, 0)$ is the triangle formed by $0, a_1$, and a_2) and let $b(a_1, a_2, a_3) = (0, a_2, a_3)$ denote the "back 1-face of T^2" (i.e., $(0, a_2, a_3)$ is the triangle formed by $0, a_2$, and a_3). Then $S \circ f$ is the imbedded front 1-face of $S(T^2)$ and $S \circ b$ is the imbedded back 1-face. Thus, it makes sense to define the cup product, $a_1 \smile b_1$, of a_1 and b_1 by the identity

$$a_1 \smile b_1[S] = (a_1[S \circ f]) \cdot (b_1[S \circ b]) \in \mathbb{Z}_2,$$

that is, we calculate the value of the 1-cochain a_1 on the 1-cycle $S \circ f$, and we multiply it (in $\mathbb{Z}_2$) by the value that the 1-cochain b_1 gives on $S \circ b$. This gives us a number in $\mathbb{Z}_2$.

Now, the problem that arises is the following: It may be (in the above described setting) that any two-cycle, $c \in H_2(M; \mathbb{Z}_2)$, satisfies the following property (property [P]):

No matter how we deform c (via a continuous deformation), it is always the case that the "front 1-face", c_1, and the "back 1-face", c_1', satisfy

$$w_1^+ \smile w_1^+[c] = (w_1^+[c_1]) \cdot (w_1^+[c_1']) = 1 \cdot 0 = 0$$

That is, it may be that we can have a cobordism M which is time-orientable ($w_1^- = 0$), is not space-orientable ($w_1^+ = 1$), and yet still satisfies $w_1^+ \smile w_1^+ = 0$!

In fact, this situation *does* occur, as seen in the following

Example 13. Let K denote the two-dimensional Klein bottle, and T^2 denote the two-dimensional torus. Then we can form a spacetime $M \simeq K \times T^2$. Remove a disk from M to obtain a spacetime $M' = M - D^4$, $\partial M' = S^3$. Then $\mathrm{kink}(\partial M'; g_{ab}) = 1$. But $u(\partial M') = 1$, and so

$$(u(\partial M') + \mathrm{kink}(\partial M'; g_{ab})) \bmod 2 = 0.$$

Now, we can always choose the Lorentz metric on M' so that M' is time-orientable but not space-orientable (signature $(-+++)$, i.e., the non-space orientability comes from the "K" part of M'), and so M' admits pin-Lorentz structure if and only $w_1^+ \smile w_1^+ = 0$, i.e., iff for any 2-cycle, c, $w_1^+ \smile w_1^+ [c] = 0$. However, K itself (viewed as a smoothly embedded 2-manifold in M') is evidently a 2-cycle (satisfying $w_1^+ \smile w_1^+ [K] = 0$), and so M' admits (global) pin-Lorentz structure, even though $w_1^+ = 1$.

Thus, we see that we can have *any* kink number in the case $w_1^+ = 1, w_1^- = 0$ depending on the topology of the cobordism, M. This is summed up in the following

Corollary 14. *Let $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be a collection of closed three-manifolds. Then there exists a pin-Lorentz cobordism, M (signature $(-+++)$), with $w_1^+(M) = 1$ and $w_1^-(M) = 0$ if and only if*

$$(u(\partial M) + \mathrm{kink}(\partial M; g_{ab})) \bmod 2 = \begin{cases} 0, & \textit{if any 2-cycle, c, satisfies Property [P]} \\ 1, & \textit{otherwise} \end{cases}$$

(theenumi) Finally, suppose we insist that our cobordism, M, be space-orientable but not time-orientable. Then $w_1^+ = 0$ and $w_1^- = 1$, and so $w_1^+ \smile w_1^+ = 0$ regardless of whether or not there is a two-cycle in M satisfying Property [P]. Thus, such a cobordism exists if and only if

$$(u(\partial M) + \mathrm{kink}(\partial M; g_{ab})) \bmod 2 = 0.$$

This exhausts all of the possibilities for signature $(-+++)$.

Case 2 (*signature* $(+---)$). In this case, we see [16] that theorem 12 is still "true"; that is, it is still the case that there exists a pin-Lorentz cobordism, M (signature $(+---)$), for $\{\Sigma_i : i = 1, ..., n\}$ if and only if the following holds:

$$(u(\partial M) + \mathrm{kink}(\partial M; g_{ab})) \bmod 2 = 0 \iff w_1^+ \smile w_1^+ = 0$$

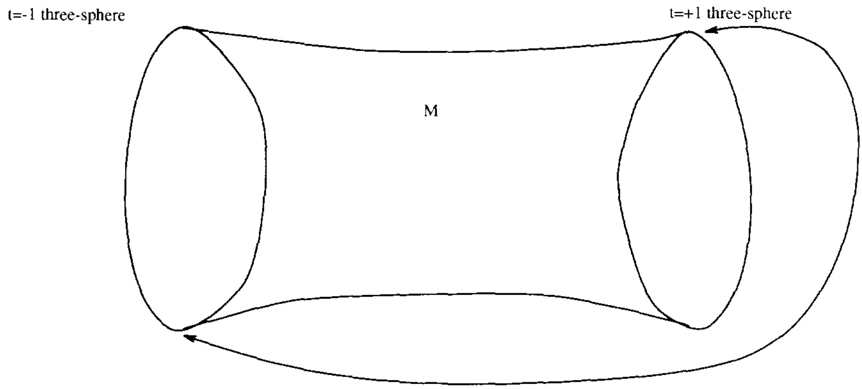

Fig. 2. Identify antipodally.

The difference now is that w_1^+ refers to the *timelike* orientation. Thus, we get the same results in *Case 2* as we did in *Case 1*, only with the values of w_1^- and w_1^+ interchanged.

We conclude with

Example 15 (Gibbons). $S^3 \times [0,1]/\mathbb{Z}_2$. Here, slice de Sitter spacetime $(+---)$ with two spacelike slices at times $t = \pm 1$, and identify the resulting three-spheres antipodally. One then obtains a space, M, which is topologically $S^3 \times [0,1]/\mathbb{Z}_2$, as shown in fig. 2 Clearly, then M is a space-orientable spacetime which is not time-orientable and has $\partial M \cong S^3$ spacelike ($\mathrm{kink}(\partial M; g_{ab}) = 0$). Hence, M has pin-Lorentz structure (since it has no 2-cycles satisfying Property [P]) and so M is the standard example of the "creation of a spacelike S^3 from nothing" spacetime (for signature $(+---)$), which we shall encounter below.

6. Applications of the obstructions

We are interested in seeing what restrictions our invariants place on the homotopy type of the metric in standard spacetime examples which are frequently encountered.

Let us first consider spin-Lorentz structure. Then one of the first examples that springs to mind is the "creation from nothing universe", which is visualised in fig. 3.

Here, M is a compact spin-Lorentz manifold with single boundary component Σ, which is to be interpreted as a "three-surface of simultaneity" with respect to some universal time indexed by a Morse function $f: M \longrightarrow \mathbb{R}$ (so that $f^{-1}(x) \cong \Sigma$, for some $x \in \mathbb{R}$).

If $\Sigma \cong S^3$, we have

$$u(\partial M) = \dim_{\mathbb{Z}_2}(H_0(S^3;\mathbb{Z}_2) \oplus H_1(S^3;\mathbb{Z}_2)) \bmod 2 = 1 \quad \Rightarrow$$

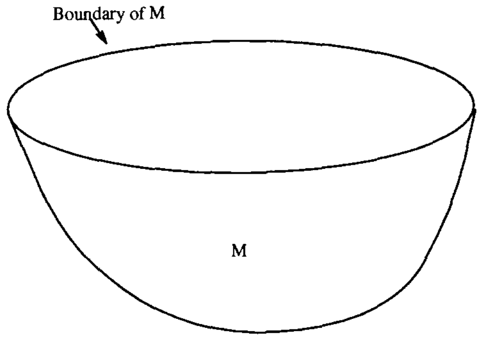

Fig. 3.

in order to have

$$(u(\partial M) + \text{kink}(\Sigma; g_{ab})) \bmod 2 = 0$$

we must have

$$\text{kink}(\Sigma; g_{ab}) = 1 \bmod 2$$

Thus, if the topology of our perceived three-surface of simultaneity Σ is S^3, then the metric g_{ab} must have non-trivial homotopy type with respect to Σ (in particular, there must exist an *odd* number of kink regions of g_{ab} with respect to Σ).

On the other hand, if $\Sigma \cong S^1 \times S^2$, then

$$u(\Sigma) = 0,$$

and so we can have a creation from nothing universe with $\text{kink}(\Sigma; g_{ab}) = 0$, as long as $\Sigma \cong S^1 \times S^2$. If we live in an expanding universe, with global spin-Lorentz structure, and our perceived three-surface of simultaneity Σ is everywhere spacelike (no kinking), then we must have $\Sigma \cong S^1 \times S^2, \mathbb{RP}^3$, or some other three-manifold satisfying

$$u(\Sigma) = 0.$$

Another example arises when one considers the creation of a single "time machine" [5], in the sense of Thorne and co-workers. Explicitly, Thorne et al. speculate that an "advanced" civilization may someday be able to create a spacetime wormhole (with spacelike topology $S^1 \times S^2$) by "pulling" such a wormhole out of the quantum foam. Thus, assuming that the initial topology (spacelike) of the universe is S^3, we are concerned with what our invariant tells us about the homotopy type of g_{ab} with respect to $S^1 \times S^2$ (the "final" topology). Writing

$$\begin{cases} \Sigma_i \cong S^3 \\ \Sigma_f \cong S^1 \times S^2 \end{cases}$$

we are concerned with the spin-Lorentz cobordism, M, for Σ_i and Σ_f. Since

$$u(\partial M) = u(\Sigma_i \cup \Sigma_f) = 1$$

we see that we must have

$$\text{kink}(\partial M) = 1 \text{ mod} 2.$$

Thus, the metric must have non-trivial homotopy on the wormhole Σ_f, assuming Σ_i was spacelike.

Now let us consider pin-Lorentz structure. Then one of the first things we notice is that we can have a creation from nothing universe, M, with $\partial M \cong S^3$ and $\text{kink}(\partial M; g_{ab}) = 0$. In other words, if the signature is $(- + ++)$ then there exist compact pin-Lorentz manifolds, which are either (i) neither space nor time-orientable, or are (ii) time-orientable but not space-orientable, and which have a single spacelike boundary component homeomorphic to S^3.

Likewise, if the signature is $(+ - --)$ then there exist compact pin-Lorentz manifolds, which are either (i) neither space nor time-orientable, or are (ii) space-orientable but not time-orientable, and which have a single boundary component homeomorphic to S^3.

Finally, let us again consider the "time machine" situation of Thorne and co-workers, i.e., $\partial M \cong S^3 \cup (S^1 \times S^2)$. Then we see that we can now have pin-Lorentz spacetimes for which both the initial slice, $\Sigma_i \cong S^3$, and the final slice, $\Sigma_f \cong S^1 \times S^2$ are *spacelike*. This is outlined as follows:

If the signature is $(- + ++)$ then there exist compact pin-Lorentz manifolds M_i, which are either (i) neither space nor time-orientable, or are (ii) time-orientable but not space-orientable, and which have everywhere spacelike boundaries $\partial M_i \cong S^3 \cup (S^1 \times S^2)$.

Following the above example, a similar statement holds for the case with signature $(+ - --)$.

We conclude with a discussion on kinking and causality.

7. Kinking and causality

Suppose M is a compact spacetime with

$$\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n \not\cong \emptyset.$$

Recently, there has been some suspicion that there may be a relation between the topology of ∂M, along with the value of $\text{kink}(\partial M; g_{ab})$, and the existence of closed timelike curves (CTCs) in M. In particular, it was conjectured [17] that if $\partial M \cong S^3$ and $\text{kink}(\partial M; g_{ab}) = 0$, then there must exist CTCs in M.

In recent work [18] with Roger Penrose, we managed to show the above conjecture to be false (by counter example); in fact we proved the more general

Theorem 16. *Let $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be any collection of closed, orientable three-manifolds, $n \in \mathbb{Z}$ an arbitrary integer. Then there exists a compact causal spacetime M with $\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$ and $\text{kink}(\partial M; v) = n$, where v is a timelike vector field.*

Theorem 17. *If M is compact and causality violating, with $\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n \ncong \emptyset$, then there exists a continuous deformation of the metric on M such that the new spacetime with deformed metric does not possess CTCs.*

Proof of theorem 16. Let $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be any collection of closed, orientable three-manifolds, $n \in \mathbb{Z}$ any integer. Then we can always find a Lorentz manifold M (with metric g_{ab} and timelike vector v) such that $\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$ and $\text{kink}(\partial M; v) = n$. This follows from the general formula (6):

$$e(M) = \sum i_v - \text{kink}(\partial M; v)$$

Now, we can cover M with a finite number of sets B_{p_i} of the form

$$B_{p_i} = \{x \in I^+(p_i) \cap I^-(q) \mid q \in I^+(p_i)\}$$

Furthermore [19], we can take the sets in this finite cover to be fine enough that they are all locally causal (i.e., no CTC lies entirely in any one of the B_{p_i}s).

Now, the crucial idea of the construction depends upon our ability to cut all of the CTCs by removing a finite number of four-balls. That we can do this is reasonably intuitively obvious, but we justify this construction more rigorously as follows.

Begin by successively removing the "$t = 0$" Cauchy surface, C_i from each of our locally causal covering sets B_{p_i}, as shown in fig. 4. Now, at each stage C_i may already be intersected by a previously removed part (assumed to be a union of three-disks), R_{i-1}, so subdivide to get a covering of what is left by three-disks, (D^3s), as shown in fig. 5. Next, modify C_i according to the two rules shown in fig. 6. Adjoin the result to R_{i-1} to get R_i, which is thus given as a disjoint union of three-balls, D^3_j, as shown in fig. 7. Finally, thicken out the D^3_js to get disjoint four-balls B^4_js which clearly cut the CTCs.

Hence, we can cut all of the CTCs with a finite number of such four-balls.

We now connect each of these little removed four-balls to the "old" boundary of $M, \partial M$, via little tubes $T_j \cong D^3 \times [0,1]$; that is, we cut out a little tube leading from some component of the "old" boundary of M to the new boundary component formed by removing a B^4_j, as shown in fig. 8

Call the new manifold obtained after such a finite sequence of operations N. Then clearly

$$\partial N \cong \partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$$

since all we did to obtain N was push a lot of "dimples" into the boundary of M (the boundary ∂N is a continuous deformation of ∂M).

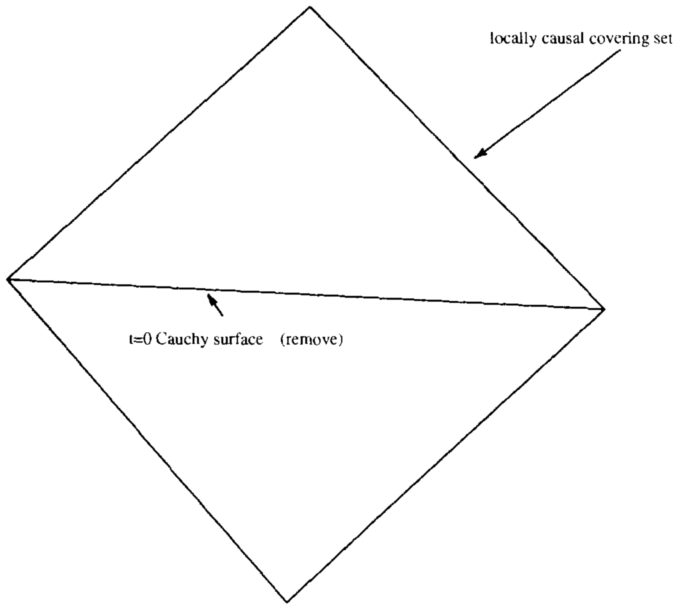

Fig. 4. The Cauchy surface we remove is homeomorphic to an open three-disk.

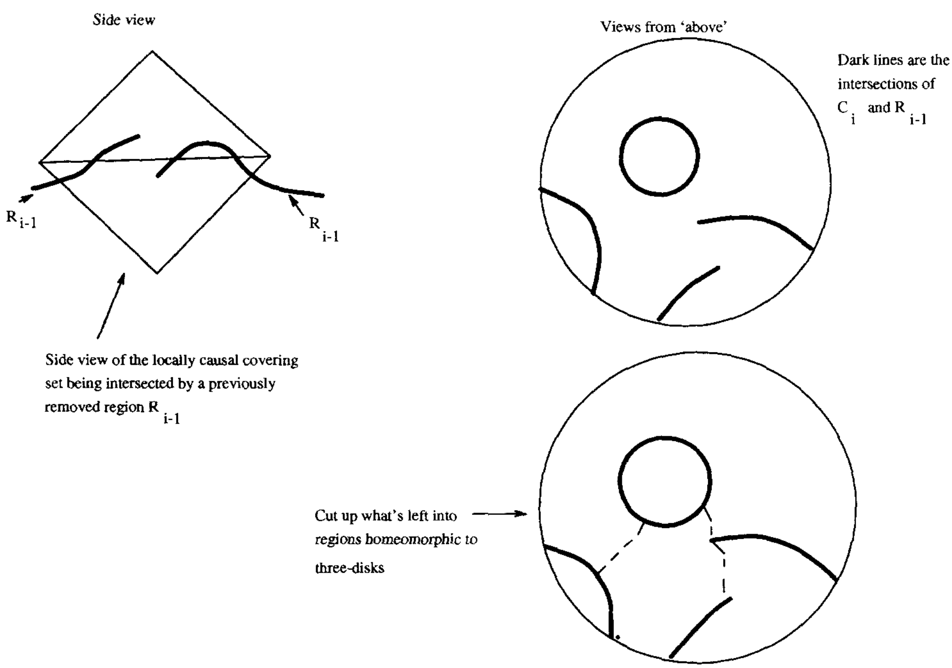

Fig. 5.

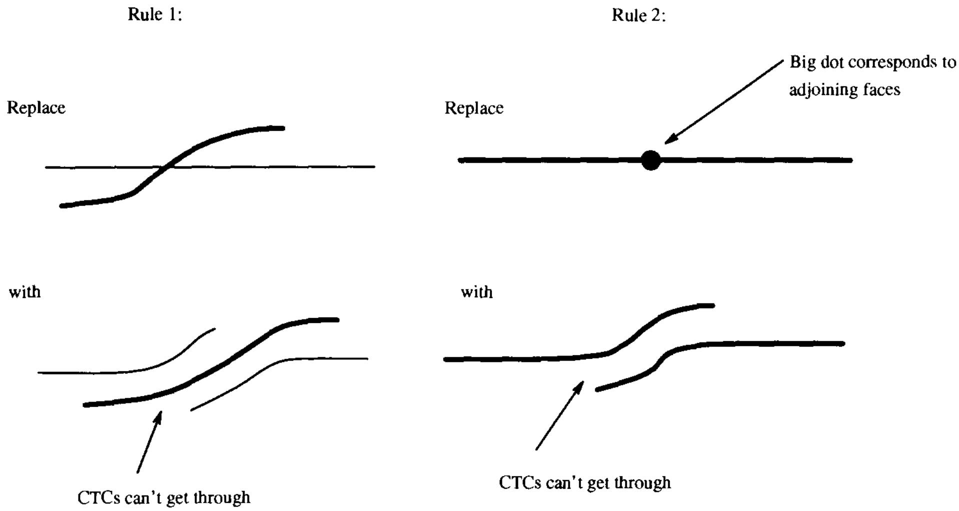

Fig. 6.

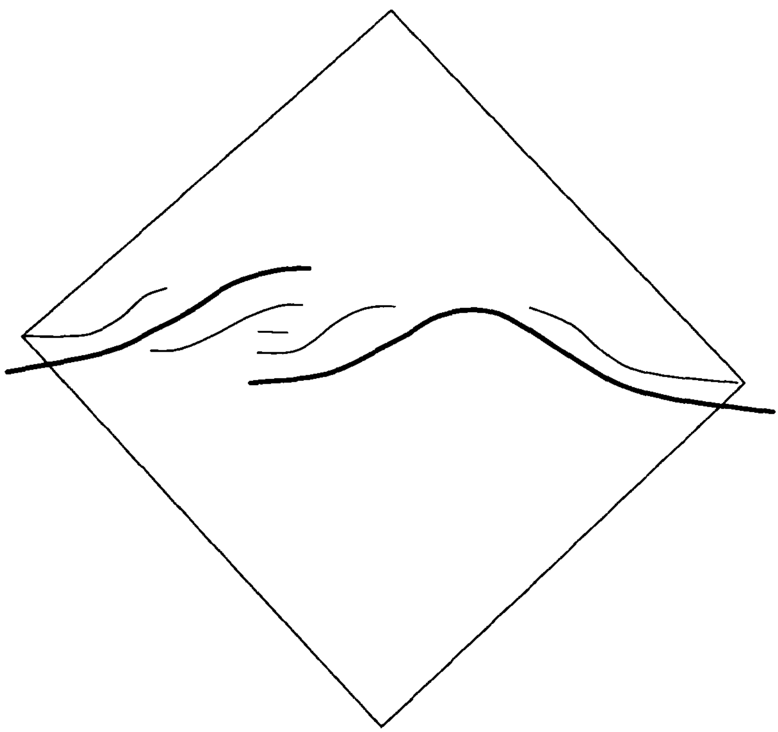

Fig. 7. R_i is the disjoint union of all the removed three-disks.

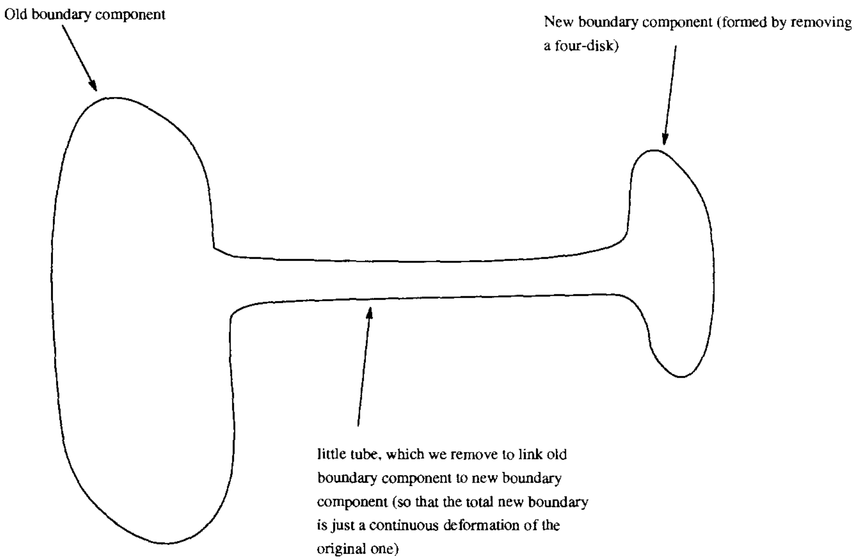

Fig. 8.

Furthermore, v (and hence g_{ab}) is still global and non-vanishing on N, i.e., $\sum i_v = 0$. Thus, $\text{kink}(\partial N; v) = n$. Thus, N is a causal spacetime with $\partial N \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$ and $\text{kink}(\partial N; g_{ab}) = n$, and the theorem is proved. □

To prove theorem 17, we continuously retract ∂N back to ∂M (via a homotopy) and "pull" the metric with the retraction (via the isotopy which lifts from the homotopy).

In closing, we note that Dr. R.P.A.C. Newman has strengthened the above proof of theorem 16 by continuously retracting ∂M all the way back to the skeleton of M. In this way, he is able to use fewer intuitive diagrams (and more theorems) to prove the result.

8. Conclusion

In closing, we point out some interesting questions which emerge from this work.

First, as was shown in [1], the obstructions to spin-Lorentz and pin-Lorentz structures can be interpreted physically as kinematical obstructions to the creation of certain types of "time machines". Are there any other kinematical aspects of physical law which one might also hope to apply to the question of the theoretical possibility of time travel (i.e., the Chronology Protection Conjecture)?

Second, does there exist any general relationship betwen kinking and geodesic incompleteness? For example, a spherically symmetric (asymptotically flat) kink region has incomplete null geodesics corresponding to the roots of g_{00} [20]. Can one develop a general statement which tells us "which" types of kinking with respect to "which" types of three-surfaces (in either compact or non-compact spacetimes) inevitably lead to geodesic incompleteness?

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